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Dynamic Investment Policy with Installation Experience Effects[1]

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Abstract. This paper analyzes the consequences of incorporating a learning-by-doing effect in the firm's adjustment cost function. The hypothesis is that, the larger the existing capital stock, the larger the installation experience gained, and therefore the smaller the cost of installing an additional unit of capital stock. The implications of the hypothesis are investigated in an optimal control model for the determination of the firm's optimal investment policy over an infinite planning period.

Key Words. Optimal control, investment policy, adjustment costs, installation experience.

1. Introduction

Adjustment costs have been introduced in a series of studies of a firm's optimal investment behavior. The assumption is that capital inputs are adjustable (quasi-fixed), but at an expense, the adjustment cost. Traditionally, a distinction has been made between internal, firm-specific adjustment costs (caused, for example, by a temporary decrease of productivity due to reorganization of the production line upon the installation of new machinery) and external adjustment costs (e.g., increasing prices of new capital goods due to monopsony in the markets for such goods).

Internal adjustment costs are incurred when resources devoted to adjustments of the capital stock have alternative use in current production. The argument is that the production function delivers jointly some output for sale and some adjustment services for internal use. Thus, the firm faces a trade-off between producing and selling more today if it would give up

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some current investment opportunities, and hence future production and sales, or it could grow faster now by investing, but at the expense of current output and sales. The cost of adjustment is the output sacrificed by devoting services to the investment process.

Formally, one can proceed as follows. Disregard labor and let $K = K(t)$ denote the stock of capital by time t. Let $I = I(t)$ denote the gross investment rate at time t. Introduce a generalized production function (Ref. 1), given by the twice differentiable function

$$Q = F(K, I), \tag{1}$$

where

$$F_I < 0, \qquad F_{II} < 0, \qquad F_K > 0, \qquad F_{KK} < 0.$$

The investment rate enters the production function with negative and decreasing marginal productivity. Capital has positive but diminishing marginal productivity.

In Ref. 1, it is assumed that F is linearly homogeneous in its arguments. Furthermore, function F is written as a sum of an ordinary production function, with argument K, and an internal adjustment cost function, with arguments I and K. Because of the homogeneity assumption, the adjustment cost function can be rewritten as a function of I/K.

Following Ref. 1, Ref. 2 states that the cost of installing I units of investment goods is likely to depend on the size of I relative to K. A separable production function $F(K) - G(I, K)$, where G is the installation function, is introduced. To install I units of capital, it is necessary to buy $G(I, K)$ units of capital. The same model is treated in Ref. 3. As in Ref. 1, the assumption of linear homogeneity yields

$$G(I, K) = g(I/K)K,$$

that is, the adjustment cost depends on the rate of accumulation rather than the level of investment. Reference 4 discusses the choice of the production function in (1) and treats a general, nonseparable specification.

The aim of this paper is to propose another motivation for the choice of an adjustment cost function having both investment and capital stock as arguments. The hypothesis probably originated in Ref. 5, where it is suggested that investment improves skills and technical knowledge such that the accumulation of installation experience is reflected in a downward drift in cost curves. Thus, investment is capable of changing the environment in which production takes place and provides new stimuli for learning. In an empirical study of US data for the period 1929–1964, Ref. 6 finds evidence to confirm that productivity growth in manufacturing is correlated with the amount of investment and output.

Remark 1.1. Alternatively, one could take cumulative production as a proxy for cumulative experience; see Refs. 5-6. The basic assumption is here that marginal costs of production decrease with increasing cumulative output (production experience).

Here, we take cumulative investment as a proxy for cumulative experience. Introduce the standard formula for net investment,

$$\dot{K} = I - aK, \qquad K(0) = K_0 > 0 \text{ and given,} \tag{2}$$

where $a > 0$ is the constant rate of depreciation. Integrating (2) shows that the measured stock of capital is

$$K(t) = \exp\{-at\}K_0 + \int_0^t \exp\{-a(t-v)\}I(v)\,dv, \tag{3}$$

and K can be interpreted to represent cumulative investment. Consider an adjustment cost function given by

$$C(I, K) = c(K)I,$$

where function c represents the unit adjustment cost. Assuming $c'(K) < 0$ means that the unit adjustment cost decreases when the capital stock increases. Cumulative installation experience is represented by the current capital stock; hence, the larger the capital stock, the smaller the cost of installing an extra unit of capital stock.

We leave aside an analysis of the investment problem with the specific adjustment cost function $c(K)I$. The investment problem becomes linear in the control variable I and can be solved as a variational problem (cf. Ref. 7).

As a more general representation of the total adjustment cost function, define a C^2 function $A(I, K)$ satisfying

$$A(I, K) \geq 0, \qquad \text{for all } I, K \geq 0, A(0, K) = 0, \tag{4a}$$

$$A_I(I, K) > 0, \qquad A_{II}(I, K) > 0, \tag{4b}$$

$$A_K(I, K) < 0, \qquad A_{KK}(I, K) > 0, \tag{4c}$$

$$A_{IK}(I, K) = A_{KI}(I, K) < 0, \tag{4d}$$

$$A_{II}(I, K)A_{KK}(I, K) > [A_{IK}(I, K)]^2. \tag{4e}$$

Assumption (4b) states that adjustment costs increase in a convex way with increasing gross investment, for a given level of K. Assumption (4c) means that adjustment costs are smaller the larger the current level of capital stock, for a given rate of gross investment. This is the learning effect hypothesis. The second part of (4c) implies that learning is subject to diminishing marginal returns. Assumption (4d) states that the increase in adjustment costs, due to an increase in gross investment, is smaller the larger the level of capital stock. Alternatively, the decrease in adjustment costs, due to learning, is smaller the larger the current gross investment rate. Finally, (4b), (4c), and (4e) imply that function A is strictly convex in (I, K).

Gross earnings of the firm are given by the revenue function $S(K)$, defined as revenue after maximization with respect to variable inputs. The revenue function is twice differentiable and satisfies

$$\begin{aligned} &S(K)>0, &&\text{for } K>0,\\ &S(0)=0, &&S'(K)>0,\\ &S'(K)\to 0, &&\text{for } K\to\infty,\\ &S''(K)<0. \end{aligned}$$

Assuming a constant price of unity for investment goods, the instantaneous profit stream becomes

$$\pi(I, K) := S(K) - I - A(I, K), \tag{5}$$

and hence

$$\pi_I < 0, \qquad \pi_{II} < 0, \qquad \pi_K > 0, \qquad \pi_{KK} < 0.$$

The signs of these derivatives correspond to those in Ref. 1.

This completes the description of the components of the dynamic investment model. The rest of the paper is organized as follows. In Section 2, the optimization problem of the firm is specified as an optimal control problem. Section 3 contains the mathematical analysis, whereas economic interpretations of the two segments of the optimal investment path can be found in Section 4. This section also contains some comparisons of the results of the present setup with those of preceding models. Section 5 concludes.

2. Optimal Control Problem

The objective functional of the firm is given by

$$J(I, K) = \int_0^\infty \exp\{-it\}\pi(I, K)\, dt, \tag{6}$$

and we assume that the integral converges for all admissible (I, K). In view of a financial constraint to be imposed [cf. (8)], the assumption does not seem unplausible. In (6), $i > 0$ denotes a constant discount rate.

The firm wishes to determine its control variable $I(t)$ so as to maximize $J(I, K)$ subject to the differential equation constraint (2) as well as

$$I \geq 0, \qquad \text{for all } t \in [0, \infty), \tag{7}$$

$$\pi(I, K) \geq 0, \qquad \text{for all admissible } (I, K). \tag{8}$$

Constraint (7) requires irreversibility of investment. Constraint (8) means that the firm cannot borrow money to finance its operations and issues of new shares are not allowed. The only source of finance is existing equity and retained earnings.

Remark 2.1. Empirically, issues of new shares have been a marginal means of finance in postwar Western economies (Ref. 8). The assumption of no borrowing is more restrictive, since in practice debt is an important source of finance. The reason for omitting the possibility of debt financing is mainly technical, but our suspicion is that the results would not change dramatically if borrowing were allowed. Under the assumptions, the firm's rate of dividend payout equals $\pi(I, K)$. Hence, constraint (8) means that the dividend rate cannot be negative. Note that a negative rate of dividend payout can be interpreted to mean that the firm issues new shares. The upshot of assumption (8) is that, during periods when desired investment would imply a negative profit rate, the firm is credit-rationed.

With the differential equation (2), it is easy to show that the natural state constraint

$$K \geq 0 \tag{9}$$

is automatically satisfied whenever $I \geq 0$.

We need to impose some regularity assumptions on the adjustment cost function A and the revenue function S. First, for all admissible K, assume that

$$\pi(aK, K) > 0, \tag{10}$$

which simply states that the firm does make a positive profit when the investment rate is at the replacement level aK. This is quite realistic. Technically, the assumption ensures that the equilibrium point is reachable under constraints (7)-(8). Second, assume that

$$S'(0) - A_K(0, 0) > (i + a)[1 + A_I(0, 0)], \tag{11}$$

which states that the marginal revenue provided by the first unit of capital exceeds the marginal cost of that unit. Notice that marginal revenue consists of two parts: the usual revenue term $S'(0)$ plus the cost saving due to increased experience [the term $-A_K(0, 0)$, which is positive]. Without installation experience, that is, without the A_K term, condition (11) is standard (e.g., Ref. 9, p. 702). Third, assume that

$$S''(K) - A_{KK}(aK, K) - (i+2a)A_{IK}(aK, K) < 0. \tag{12}$$

This condition does not seem to have a meaningful economic interpretation. Technically, it is needed to establish the equilibrium as a saddle point. Note that, if function A is separable in I and K, (12) is trivially satisfied.

3. Analysis of the Control Problem

Define the current value Hamiltonian

$$H(I, K, m, m_0) = m_0 \pi(I, K) + m(I - aK),$$

where $m_0 \geq 0$ is a constant and $m = m(t)$ is a current value adjoint variable. Define the Lagrangian

$$L(I, K, m, \mu_1, \mu_2) = H(K, I, m) + \mu_1 \pi(I, K) + \mu_2 I,$$

where $\mu_i = \mu_i(t)$, $i = 1, 2$, are Lagrange multiplier functions.

We make two preliminary observations:

(a) using (4b), (4c), (4e), and $S'' < 0$ shows that H is strictly concave in (I, K) for each t;

(b) functions I and $\pi(I, K)$ [that is, the left-hand sides of constraints (7) and (8)] are concave in (I, K) for each t.

The following sufficient optimality conditions appear in Refs. 10 and 11. Let the pair (I^*, K^*) be feasible, that is, it satisfies (2), (7), (8), and (9). Assume the existence of a continuous and piecewise continuously differentiable function $m = m(t)$ and piecewise continuous functions $\mu_1(t)$, $\mu_2(t)$, such that the following conditions are fulfilled with $m_0 = 1$:

$$L_I = -[1 + A_I(I^*, K^*)](1 + \mu_1) + m + \mu_2 = 0, \tag{13a}$$

$$\mu_1 \geq 0, \qquad \mu_1 \pi(I^*, K^*) = 0, \tag{13b}$$

$$\mu_2 \geq 0, \qquad \mu_2 I^* = 0, \tag{13c}$$

$$\dot{m} = (i + a)m - (1 + \mu_1)[S'(K^*) - A_K(I^*, K^*)], \tag{13d}$$

$$\lim_{t \to \infty} \exp\{-it\} m(t)[K(t) - K^*(t)] \geq 0 \qquad \text{for all feasible } K(t). \tag{13e}$$

Then, because of the observations (a) and (b), the pair (I^*, K^*) is an optimal solution.

Remark 3.1. Recall that $K \geq 0$ is guaranteed for all t. The limiting transversality condition (13e) is satisfied if the following conditions are met for all feasible $K(t)$:

(i) $m(t) \geq 0$ and bounded for all t;
(ii) $K^*(t)$ is bounded for all t.

Remark 3.2. No constraint qualification was included in the above sufficiency conditions. However, the multipliers will normally satisfy the requirements stated above only if a constraint qualification is fulfilled; otherwise, one must allow the costate $m(t)$ to be piecewise continuous. With respect to the problem at hand, consider the constraints (7) and (8) and the associated matrix

$$M = \begin{bmatrix} 1 & I & 0 \\ -1 - A_I(I, K) & 0 & \pi(I, K) \end{bmatrix}.$$

A constraint qualification requires that M have row rank equal two. The only case in which this may fail to hold is when both constraints (7) and (8) are binding on a nonzero interval. However, for such a case to occur, it must be true that $S(K) = 0$, and hence $K = 0$, on the interval. We claim that this can be true only in the limit.

To prove the claim, recall the constraint $I(t) \geq 0$ for $t \in [0, \infty)$, and consider the policy $I^*(t) = 0$ for $t \in (0, \infty)$. Using (3), we find the associated capital stock to be

$$K^*(t) = K_0 \exp\{-at\}.$$

Since K_0 is positive, it holds that $K^*(t) > 0$ for all $t < \infty$ and $K^*(t) \to 0$ for $t \to \infty$. However, using (3) once more shows that the capital stock $K(t)$ associated with any feasible policy $I(t)$, not identical to $I^*(t)$, satisfies

$$K(t) > K^*(t).$$

The next step is to investigate the optimality conditions in (13). By (13a), the investment rate I is implicitly determined as a function of K and m (as well as μ_1, μ_2). Let this function be denoted by

$$I = g(K, m; \mu_1, \mu_2)$$

and notice that

$$g_K = -A_{IK} / A_{II} > 0.$$

The economic implication is that, *ceteris paribus*, the larger the existing capital stock the larger the investment rate. In the case of no learning effects, we would have $A_{IK} = 0$, implying $g_K = 0$, that is, the investment rate does not depend on the actual amount of capital goods. Moreover,

$$g_m = 1/(1 + \mu_1) A_{II} > 0.$$

This is what would be expected; the larger the shadow price m of the capital stock, the larger the investment rate.

Using (13a)–(13c) provides a characterization of three possible investment paths.

Maximal Investment Path. It holds that

$$\pi(I, K)=0, \qquad \mu_1 \geq 0, \qquad I>0, \qquad \mu_2=0,$$

$$m=(1+A_I)(1+\mu_1).$$

The assumption in (10) guarantees that K increases on this path. Solving the equation $\pi(I, K)=0$ with respect to I for any feasible K yields implicitly $I=\Phi(K)$. The function Φ is increasing, since

$$\Phi'=-\pi_K/\pi_I=(1+A_I)^{-1}[S'(K)-A_K]>0.$$

This means that the growth process is self-increasing: as the stock of capital goods increases, the larger stock levels permit higher rates of maximal investment that, in turn, raise the stock even more. For an analogous result in a linear model, see Ref. 7. The investment rate is also determined as

$$I=g(K, m, \mu_1, 0).$$

Interior Investment Path. It holds that

$$I>0, \; \pi(I, K)>0, \qquad \mu_1=\mu_2=0, \qquad m=1+A_I.$$

The investment rate is determined by

$$I=g(K, m, 0, 0).$$

Zero Investment Path. It holds that

$$I=0, \qquad \mu_2 \geq 0, \qquad \pi(I, K)>0, \qquad \mu_1=0,$$

$$m=1+A_I-\mu_2.$$

The investment rate is determined by

$$I=g(K, m, 0, \mu_2).$$

Remark 3.3. Maximization of the Hamiltonian subject to (7) and (8) yields

$$I=\begin{cases}\text{maximal}, & m>1+A_I,\\ \text{interior}, & m=1+A_I,\\ \text{zero}, & m<1+A_I.\end{cases}$$

The interpretation of the occurrence of each investment path is straightforward, since m represents the shadow price of a unit of capital stock and the term $1+A_I$ is the expense incurred by buying and installing an extra unit of capital.

Use (13a) to rewrite (13d) as

$$\dot{m} = -(1+\mu_1)[S'(K) - A_K] + (i+a)[(1+A_I)(1+\mu_1) - \mu_2], \tag{14}$$

and consider interior points satisfying

$$I > 0 \quad \text{and} \quad \pi(K, I) > 0.$$

By (13b), (13c),

$$\mu_1 = \mu_2 = 0.$$

At points of differentiability of I, we get using (13a)

$$\dot{m} = A_{II}\dot{I} + A_{IK}\dot{K}; \tag{15}$$

and substitution from (2) and (14) into (15) yields

$$\dot{I} = [A_{II}]^{-1}[(i+a)(1+A_I) - S'(K) + A_K - A_{IK}(I - aK)]. \tag{16}$$

We wish to study the differential equation system (2) and (16) in the (K, I) phase plane and restrict our analysis to equilibria (K_∞, I_∞) located in the interior of the control region defined by (7) and (8). If an equilibrium point exists it must satisfy

$$I_\infty = aK_\infty, \tag{17a}$$

$$(i+a)[1 + A_I(I_\infty, K_\infty)] = S'(K_\infty) - A_K(I_\infty, K_\infty). \tag{17b}$$

The economic interpretation of the equilibrium point is as follows. The investment rate is maintained at the constant replacement level aK_∞, and marginal revenue [right-hand side of (17b)] equals marginal cost [left-hand side of (17b)]. Marginal revenue consists of two parts: the usual term S' and the cost saving $-A_K$ due to increased experience. Without the A_K term, the expression (17b) becomes standard (e.g., Ref. 9, pp. 698–699).

The elements of the Jacobian matrix are

$$\delta\dot{K}/\delta K = -a, \tag{18a}$$

$$\delta\dot{K}/\delta I = 1, \tag{18b}$$

$$\delta\dot{I}/\delta K|_{(K_\infty, I_\infty)} = [A_{II}]^{-1}[(i+2a)A_{IK} - S''(K) + A_{KK}], \tag{18c}$$

$$\delta\dot{I}/\delta I|_{(K_\infty, I_\infty)} = i + a, \tag{18d}$$

and the Jacobian determinant, evaluated at the equilibrium, equals

$$D = -a(i+a) - [A_{II}]^{-1}[(i+2a)A_{IK} - S''(K) + A_{KK}]. \tag{19}$$

The $\dot{K}=0$ isocline is given by

$$I = aK.$$

The $\dot{I}=0$ isocline is given by

$$(i+a)(1+A_I) - S'(K) + A_K - A_{IK}(I-aK) = 0, \tag{20}$$

with slope in a neighborhood of the equilibrium

$$\begin{aligned} dI/dK|_{(K_\infty, I_\infty)} &= -(\delta\dot{I}/\delta K)/(\delta\dot{I}/\delta I) \\ &= -[(i+2a)A_{IK} - S''(K) + A_{KK}]/(i+a)A_{II}. \end{aligned} \tag{21}$$

This slope is negative because of the assumption made in (12). The $I = aK$ isocline obviously has positive slope. Rewriting (19) shows that

$$D = (i+a)[-a + dI/dK],$$

where dI/dK is given by (21). Hence, D is negative and an equilibrium point in the (K, I) phase plane must be a saddle point.

We need to verify the existence of an equilibrium in the interior of the control region given by (7)–(8). Consider the isocline in (20) and put $K=0$. Assuming that $I=0$ solves (20) leads to a contradiction with the assumption made in (11). Hence, the isocline must admit a positive value to I at $K=0$. A similar argument shows, on the same isocline, that K must have a positive value when $I=0$. It follows that $I_\infty > 0$, $K_\infty > 0$. The assumption made in (10) guarantees that $\pi(aK_\infty, K_\infty) > 0$.

The stable path to the equilibrium is declining, at least in a neighborhood of the equilibrium. This follows from the fact that the equilibrium is a saddle; one isocline has slope equal to a, the other one has negative slope.

Now, we turn to a characterization of the curve $\pi(I, K) = 0$ [cf. constraint (8)], which defines the maximal value of I implicitly as a function of K, $I = \Phi(K)$. As already demonstrated for the maximal investment path, function Φ is increasing in K. Moreover, putting $\pi(I, 0) = -I - A(I, 0)$ equal to zero implies $I=0$, that is, function Φ starts at the origin.

Remark 3.4. The second-order derivative of Φ is given by

$$\Phi'' = [1+A_I]^{-1}[S''(K) - A_{KK} - 2A_{IK}\Phi' - A_{II}(\Phi')^2],$$

the sign of which cannot be ascertained without introducing further assumptions (which we do not wish to do since their economic interpretations are, at best, dubious). However, it is free to conjecture that $\Phi'' < 0$, making function Φ concave.

Recall that

$$\Phi'(K) = (1 + A_I)^{-1}[S'(K) - A_K] > 0,$$

and hence

$$\Phi'(0) = [1 + A_I(0, 0)]^{-1}[S'(0) - A_K(0, 0)].$$

Note that $I = 0$ for $K = 0$ on $\pi(I, K) = 0$. The assumption in (11) implies $\Phi'(0) > a$, where a is the slope of the isocline $I = aK$. Also, it holds by the assumption in (10) that $\pi(aK, K) > 0$, i.e., $aK < \Phi(K)$. We conclude that, for all $K \in (0, \infty)$, the isocline $I = aK$ lies below the upper boundary $I = \Phi(K)$ of the control region.

Remark 3.5. On the curve $\pi(I, K) = 0$, the time derivatives of I and K have the same sign. To see this, differentiate totally with respect to time and obtain

$$(S'(K) - A_K)\dot{K} = (1 + A_I)\dot{I}.$$

The result follows, since

$$S' > 0, \qquad A_K < 0, \qquad A_I > 0.$$

In economic terms, when investment is at its maximal rate, investment and capital stock both increase over time, since the capital stock increases when the maximal investment path is followed.

Figure 1 shows the phase diagram. If the stable path to the equilibrium can be extended that far to the left of the equilibrium point such that the path intersects the boundary given by $\pi(I, K) = 0$, let K_1 denote the value of K for which the intersection takes place. Recall, however, that we were only able to show that the $\dot{I} = 0$ isocline was decreasing in a neighborhood of the equilibrium. But if one is willing to assume that this neighborhood includes the rectangular area given by $[I_\infty, I_2] \times [K_1, K_\infty]$, then the global saddle point theorem (e.g., Ref. 11, p. 116) assures the existence of a unique, monotonously decreasing stable path from (K_1, I_1) to (K_∞, I_∞).

Consider the economically most plausible case where the firm starts out with a small initial capital stock $K_0 < K_1 < K_\infty$. In this case, the firm would invest at the maximal rate $I = \Phi(K)$, as long as K is smaller than K_1. When K_1 is reached, the investment policy switches to the interior policy, which will imply that (K, I) approaches (K_∞, I_∞) along the stable path, as time goes to infinity.

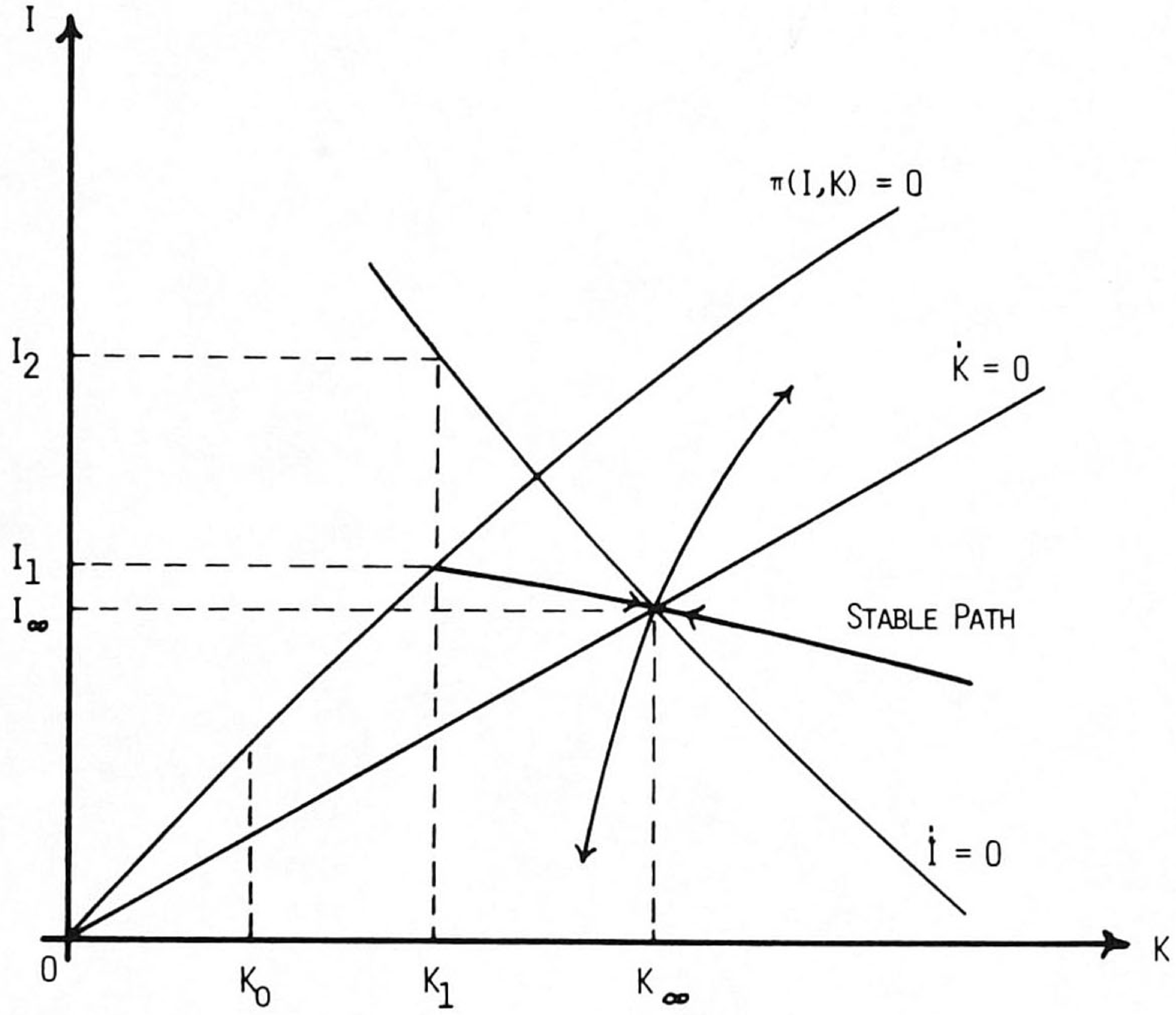

Fig. 1. Phase diagram in the (K, I) plane: $K_0 < K_\infty$.

Remark 3.6. If $K_0 > K_1$ but $K_0 < K_\infty$, the firm should use the interior investment policy as of time zero. The constants resulting from integrating the differential equations (2) and (16) can be determined such that $(K_0, I(0))$ is located on the stable path to the equilibrium. Thus, the optimal policy is simply to stay on that stable path for $t \in [0, \infty)$.

Based on the preceding analysis, we propose that the following policy is optimal:

For $t \in [0, \tau]$ put $I = \Phi(K)$. The instant τ is the time at which the capital stock reaches the level K_1. For $t > \tau$, use the interior investment policy.

What is left to prove is the optimality of the maximal investment path on the time interval $[0, \tau]$. First, observe that, on $[0, \tau]$, both the capital stock and the investment rate increases (cf. Remark 3.5).

On the time interval (τ, ∞), the investment rate evolves over time according to the differential equation (16). The slope of the stable branch is given by

$$
\begin{aligned}
dI/dK &= \dot{I}/\dot{K} \\
&= (A_{II})^{-1}[(i+a)(1+A_I) - S'(K) + A_K - A_{IK}(I - aK)]/\dot{K},
\end{aligned}
$$

which is negative (K increases but I decreases). The slope of the function defining the upper boundary of the control region, that is, $I = \Phi(K)$, is given by

$$\Phi'(K) = (1 + A_I)^{-1}[S'(K) - A_K].$$

This slope is positive. Hence, for $t = \tau$, it holds that

$$\begin{aligned}&(1 + A_I)^{-1}[S'(K) - A_K]\\&> (A_{II})^{-1}[(i + a)(1 + A_I) - S'(K) + A_K - A_{IK}(I - aK)]/\dot{K},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&[(S'(K) - A_K)A_{II} + (1 + A_I)A_{IK}]\dot{K}\\&> [(i + a)(1 + A_I) - S'(K) + A_K](1 + A_I).\end{aligned} \tag{22}$$

On an interval to the left of τ, $[\tau - \epsilon, \tau]$ say, we know that [cf. (13a)]

$$(1 + A_I)(1 + \mu_1) - m = 0 \leftrightarrow \mu_1 = -1 + m/(1 + A_I). \tag{23}$$

Assuming that μ_1 is differentiable in a neighborhood of τ, we differentiate with respect to time in (23). In the left-hand limit τ^-, we get

$$\dot{\mu}_1(\tau^-) = (1 + A_I)^{-2}[(1 + A_I)\dot{m}(\tau^-) - m(\tau^-)(A_{II}\dot{I}(\tau^-) + A_{IK}\dot{K}(\tau^-))], \tag{24}$$

where the derivatives of function A are evaluated at $t = \tau^-$.

On the right-hand side of (24), insert the following: the right-hand side of the adjoint equation (14) with $\mu_2 = 0$, the relation stated in Remark 3.5, and (23). This yields

$$\begin{aligned}\dot{\mu}_1(\tau^-) = \{&-(1 + \mu_1)[(1 + A_I)A_{IK} + A_{II}(S'(K) - A_K)]\dot{K}\\&+(1 + \mu_1)(1 + A_I)[(i + a)(1 + A_I) - S'(K) + A_K]\}[1 + A_I]^{-2}.\end{aligned} \tag{25}$$

Using (22) in (25) shows that

$$\dot{\mu}_1(\tau^-) < 0,$$

and μ must be positive on the interval $[\tau - \epsilon, \tau]$. We conclude that the optimality conditions in (13a)-(13d) are fulfilled for the boundary solution on the interval $[\tau - \epsilon, \tau]$.

The question remains whether we can put $\tau - \epsilon = 0$, that is, the optimality conditions will be fulfilled for all $K < K_1$ starting in K_0. However, the result just derived may hold only in a neighborhood to the left of K_1, and hence we cannot guarantee optimality of a maximal investment path starting at a K_0 which is far below K_1. But, for K_0 sufficiently close to K_1, optimality is obtained.

Viewing this comment in association with the fact that we needed to extend the decline of isocline (20) to the left of the equilibrium shows that the results that we have obtained should not be interpreted to hold globally. Obviously, in cases where $K_0 \approx K_1 \approx K_\infty$, we can be sure that the policy is optimal.

4. Economic Analysis

Consider the stable path to the equilibrium, i.e., the interior investment policy. From (13a) and (13d), we derive the following condition which holds for each t:

$$0 = -1 - A_I(I(t), K(t)) + \int_t^\infty \exp\{-(i+a)(s-t)\}[S'(K(s)) - A_K(I(s), K(s))]\, ds, \tag{26}$$

where the right-hand side is the net present value of marginal investment. For an interpretation, consider the acquisition of an extra unit of capital at time t. The firm incurs an extra expense at time t in amount of 1 (acquisition) plus A_I (installation). On the other hand, the marginal unit of capital generates as of time t a stream of revenue (ordinary revenue S' plus cost saving A_K). This stream is corrected for depreciation by multiplication by $\exp\{-a(s-t)\}$ and is discounted to time t by multiplication by $\exp\{-i(s-t)\}$.

Condition (26) then states that the net present value of marginal investment equals zero on the optimal path to the equilibrium. Hence, on this path, the fundamental economic principle of balancing marginal revenues with marginal expenses applies.

Next, consider the path of maximal investment which is employed during an initial interval of time $[0, \tau]$. With $\mu_1 > 0$, one obtains for each t

$$\begin{aligned}&\mu_1(t)[1 + A_I(I(t), K(t))]\\ &= -1 - A_I(I(t), K(t))\\ &+ \int_t^\infty \exp\{-(i+a)(s-t)\}[S'(K(s)) - A_K(I(s), K(s))][1+\mu_1(s)]\, ds.\end{aligned} \tag{27}$$

On the right-hand side of (27), the term $-1 - A_I$ again represents marginal expenses and the integral represents marginal revenues. The left-hand side of (27) is positive, which implies that, on this path, marginal revenue exceeds marginal expenses at each instant t. This provides the right incentive for the firm to apply maximal investment efforts.

To interpret the integral on the right-hand side, recall that μ_1 is the Lagrangian multiplier associated with the constraint

$$\pi(I, K) = S(K) - I - A(I, K) \geq 0.$$

This constraint puts an upper bound on investment in the sense that current investment outlays cannot exceed a limit dictated by the requirement $\pi(I, K) \geq 0$. The multiplier μ_1 represents the increase in the optimal objective function, caused by a one dollar increase in the upper bound. The acquisition of an additional unit of capital at time t increases the upper bound, at each instant $s > t$, by the amount

$$[S'(K(s)) - A_K(I(s), K(s))] \exp\{-a(s-t)\}.$$

The shadow price per unit such increase, evaluated in dollars at time t, is

$$\mu_1 \exp\{-i(s-t)\}.$$

Multiplying these terms produces

$$[S'(K(s)) - A_K(I(s), K(s))] \exp\{-(i+a)(s-t)\}\mu_1,$$

representing the indirect marginal earnings produced by a slight increase in the upper bound on investment. Add to this the direct marginal earnings

$$[S'(K(s)) - A_K(I(s), K(s))] \exp\{-(i+a)(s-t)\},$$

as already encountered in (26).

The next task is to compare the optimal policy under installation experience to former results that hold when there are no learning effects in the adjustment cost function. The comparison will be carried out in terms of steady-state capital stocks and investment rates.

Remark 4.1. The constraint (8) is not that important here; also, without installation experience, an optimal policy normally starts with a phase of maximal investment effort. See Ref. 12.

Consider (17), which determines the equilibrium capital stock K_∞ and the corresponding investment rate $I_\infty = aK_\infty$. We have [see (17b)]

$$S'(K_\infty) = i + a + (i+a)A_I(aK_\infty, K_\infty) + A_K(aK_\infty, K_\infty).$$

Case I. If there are no adjustment costs at all, the equilibrium capital stock, K_n say, satisfies the well-known marginal productivity formula

$$S'(K_n) = i + a. \tag{28}$$

Case II. If there are adjustment costs, but these costs depend only on the investment rate, the adjustment cost function is given by $C(I)$, and the steady-state stock (K_S say) is given by

$$S'(K_s) = i + a + (i+a)C'(aK_s); \tag{29}$$

see, for example, Ref. 12.

Comparing (28) and (29) yields the obvious result that, due to the presence of an adjustment cost, the level K_s is less than K_n.

When comparing (17b) and (28), notice that $A_I > 0$ but $A_K < 0$. However, we can obtain

$$K_\infty \lesseqqgtr K_n, \qquad \text{if } (i+a)A_I + A_K \lesseqqgtr 0, \tag{30}$$

where both partial derivatives are evaluated at the equilibrium (I_∞, K_∞). Define the following (partial) elasticities with respect to the adjustment cost function $A(I, K)$:

$$e_I = (I/A)A_I > 0, \qquad e_K = (K/A)A_K < 0,$$

where I and K are at the equilibrium values. Inserting into (30) yields

$$K_\infty \lesseqqgtr K_n, \qquad \text{if } 1 + ia^{-1} \gtreqqless -e_K/e_I. \tag{31}$$

Consider a situation where the firm is relatively farsighted such that the discount rate i is small. In this case, (31) shows that the equilibrium stock K_∞ is greater than K_n if the elasticity of the adjustment cost function with respect to K numerically exceeds the elasticity with respect to I. Thus, if installation experience should be able to counterbalance the direct increase in adjustment costs due to additional investment, the adjustment cost function must be more elastic with respect to the capital stock than the investment rate.

To compare K_∞ and K_s, recall that it was established that $K_s < K_n$. Hence, whenever K_∞ exceeds K_n [lower inequality in (31)], it also exceeds K_s. A direct comparison of (17b) and (29) yields

$$K_\infty \lesseqqgtr K_s, \qquad \text{if } (i+a)A_I(aK_\infty, K_\infty) + A_K(aK_\infty, K_\infty) \lesseqqgtr (i+a)C'(aK_s). \tag{32}$$

To evaluate (32), we need to assume that the adjustment cost function $A(I, K)$ is additive such that

$$A(I, K) = C(I) + G(K),$$

where $C(I)$ is the function employed in Case II ($C' > 0$) and $G' < 0$. Thus,

$$A_I(I, K) = C'(I),$$

and (32) becomes

$$K_\infty \lesseqqgtr K_s, \qquad \text{if } (i+a)C'(aK_\infty) + G'(K_\infty) \lesseqqgtr (i+a)C'(aK_s),$$

which is equivalent to

$$K_\infty \lesseqqgtr K_s, \qquad \text{if } (i+a)[C'(aK_\infty) - C'(aK_s)] \gtreqqless -G'(K_\infty). \tag{33}$$

Note that in (33) neither the upper inequalities nor the equalities can be true.

To summarize, the equilibrium stock K_∞, of capital under installation experience, exceeds the equilibrium stock K_s, in the absence of installation experience in at least the following two cases:

(i) A low discount rate in combination with an adjustment cost function that is more elastic with respect to capital stock (i.e., installation experience) than with respect to investment rate.
(ii) An additive adjustment cost function.

5. Concluding Remarks

This paper has studied the standard dynamic investment model of a firm, extended with installation experience in the adjustment cost function. Furthermore, an endogenous upper bound on the investment rate was imposed. The results were derived by using optimal control theory and were economically interpreted in terms of the concept of net present value of marginal investment. Comparisons with previous results were made.

The financial side of the investment problem was rather neglected in our model, since it was assumed that internal funds were the only source of additional finance. With respect to investment, the firm could only invest in capital goods. Suggested extensions would incorporate the possibility of taking loans as a means of finance and allowing the firm to lend money (i.e., making financial investments) as a second investment alternative.

References

1. LUCAS, R. E., *Adjustment Costs and the Theory of Supply*, Journal of Political Economy, Vol. 75, pp. 321–334, 1967.
2. HAYASHI, F., *Tobin's Marginal q and Average q: A Neoclassical Interpretation*, Econometrica, Vol. 50, pp. 213–224, 1982.
3. D'AUTUME, A., and MICHEL, P., *Future Investment Constraints Reduce Present Investment*, Econometrica, Vol. 53, pp. 203–206, 1985.
4. TREADWAY, A. B., *Adjustment Costs and Variable Inputs in the Theory of the Competitive Firm*, Journal of Economic Theory, Vol. 2, pp. 329–347, 1970.
5. ARROW, K. J., *The Economic Implications of Learning by Doing*, Review of Economic Studies, Vol. 29, pp. 155–173, 1962.
6. SHESHINSKI, E., *Tests of the Learning-by-Doing Hypothesis*, Review of Economics and Statistics, Vol. 49, pp. 568–578, 1967.

7. JØRGENSEN, S., and KORT, P. M., *Optimal Dynamic Investment with Learning by Doing in the Adjustment Cost Function*, to appear in Zeitschrift für Operations Research.
8. SINN, H. W., *Capital Income Taxation and Resource Allocation*, North-Holland, Amsterdam, Holland, 1987.
9. TAKAYAMA, A., *Mathematical Economics*, 2nd Edition, Cambridge University Press, Cambridge, England, 1985.
10. SEIERSTAD, A., and SYDSAETER, K., *Optimal Control Theory with Economic Applications*, North-Holland, Amsterdam, Holland, 1987.
11. FEICHTINGER, G., and HARTL, R. F., *Optimale Kontrolle Ökonomischer Prozesse*, Walter de Gruyter, Berlin, Germany, 1986.
12. KORT, P. M., *A Dynamic Net Present Value Rule in a Financial Adjustment Cost Model*, Optimal Control Applications and Methods, Vol. 11, pp. 277-282, 1990.